The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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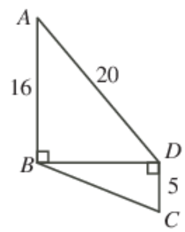
Topic Generator - Solution Set
Solutions

1. In the diagram, what is the length of BC?



(A) 13 (B) 12 (C) 20 (D) 16 (E) 17

Source: 2006 Gauss Grade 8 #16

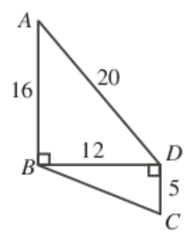
Primary Topics: Geometry and Measurement

Secondary Topics: Triangles | Measurement | Pythagorean Theorem

Answer: A

Solution:

By the Pythagorean Theorem in $\triangle ABD$, we have $BD^2 + 16^2 = 20^2$ or $BD^2 + 256 = 400$ or $BD^2 = 144$. Therefore, $BD = 12$.



By the Pythagorean Theorem in $\triangle BDC$, we have $BC^2 = 12^2 + 5^2 = 144 + 25 = 169$, so $BC^2 = 169$ or $BC = 13$.

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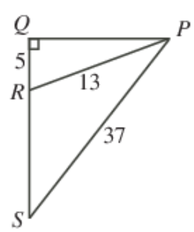
2. In the diagram, what is the perimeter of $\triangle PQS$?



(A) 74 (B) 55 (C) 80 (D) 84 (E) 97

Source: 2007 Pascal Grade 9 #18

Primary Topics: Geometry and Measurement

Secondary Topics: Perimeter | Triangles | Pythagorean Theorem

Answer: D

Solution:

By the Pythagorean Theorem in $\triangle PQR$, $PQ^2 = PR^2 - QR^2 = 13^2 - 5^2 = 144$, so $PQ = \sqrt{144} = 12$.
By the Pythagorean Theorem in $\triangle PQS$, $QS^2 = PS^2 - PQ^2 = 37^2 - 12^2 = 1225$, so $QS = \sqrt{1225} = 35$.
Therefore, the perimeter of $\triangle PQS$ is $12 + 35 + 37 = 84$.

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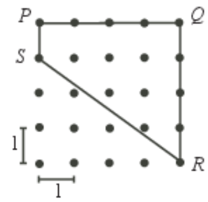
3. In the diagram, the horizontal distance between adjacent dots in the same row is 1. Also, the vertical distance between adjacent dots in the same column is 1. What is the perimeter of quadrilateral $PQRS$?



(A) 12 (B) 13 (C) 14 (D) 15 (E) 16

Source: 2012 Pascal Grade 9 #14

Primary Topics: Geometry and Measurement

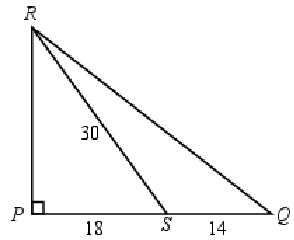
Secondary Topics: Perimeter | Quadrilaterals | Pythagorean Theorem

Answer: C

Solution:

The perimeter of quadrilateral $PQRS$ equals $PQ + QR + RS + SP$.
Since the dots are spaced 1 unit apart horizontally and vertically, then $PQ = 4$, $QR = 4$, and $PS = 1$.
Thus, the perimeter equals $4 + 4 + RS + 1$ which equals $RS + 9$.
We need to determine the length of RS.
If we draw a horizontal line from S to point T on QR, we create a right-angled triangle STR with $ST = 4$ and $TR = 3$.
By the Pythagorean Theorem, $RS^2 = ST^2 + TR^2 = 4^2 + 3^2 = 25$.
Since $RS > 0$, then $RS = \sqrt{25} = 5$.
Thus, the perimeter of quadrilateral $PQRS$ is $5 + 9 = 14$.

4. In $\triangle PQR$, $\angle RPQ = 90^\circ$ and S is on PQ. If $SQ = 14$, $SP = 18$, and $SR = 30$, then the area of $\triangle QRS$ is



(A) 84 (B) 168 (C) 210 (D) 336 (E) 384

Source: 2014 Pascal Grade 9 #15

Primary Topics: Geometry and Measurement

Secondary Topics: Triangles | Area | Pythagorean Theorem

Answer: B

Solution:

Solution 1
Since $\triangle RPS$ is right-angled at P, then by the Pythagorean Theorem, $PR^2 + PS^2 = RS^2$ or $PR^2 + 18^2 = 30^2$.
This gives $PR^2 = 30^2 - 18^2 = 900 - 324 = 576$, from which $PR = 24$, since $PR > 0$.
Since P, S and Q lie on a straight line and RP is perpendicular to this line, then RP is actually a height for $\triangle QRS$ corresponding to base SQ.
Thus, the area of $\triangle QRS$ is $\frac{1}{2}(24)(14) = 168$.

Solution 2
Since $\triangle RPS$ is right-angled at P, then by the Pythagorean Theorem, $PR^2 + PS^2 = RS^2$ or $PR^2 + 18^2 = 30^2$.
This gives $PR^2 = 30^2 - 18^2 = 900 - 324 = 576$, from which $PR = 24$, since $PR > 0$.
The area of $\triangle QRS$ equals the area of $\triangle RPQ$ minus the area of $\triangle RPS$.
Since $\triangle RPQ$ is right-angled at P, its area is $\frac{1}{2}(PR)(PQ) = \frac{1}{2}(24)(18+14) = 12(32) = 384$.
Since $\triangle RPS$ is right-angled at P, its area is $\frac{1}{2}(PR)(PS) = \frac{1}{2}(24)(18) = 12(18) = 216$.
Therefore, the area of $\triangle QRS$ is $384 - 216 = 168$.

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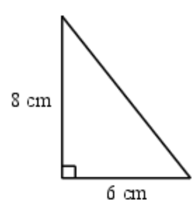
5. There is a square whose perimeter is the same as the perimeter of the triangle shown. The area of that square is



(A) 12.25 cm^2 (B) 196 cm^2 (C) 49 cm^2 (D) 36 cm^2 (E) 144 cm^2

Source: 2015 Gauss Grade 8 #16

Primary Topics: Geometry and Measurement

Secondary Topics: Triangles | Area | Perimeter | Pythagorean Theorem

Answer: D

Solution:

First, we must determine the perimeter of the given triangle.

Let the unknown side length measure x cm.

Since the triangle is a right-angled triangle, then by the Pythagorean Theorem we get $x^2 = 8^2 + 6^2$ or $x^2 = 64 + 36 = 100$ and so $x = \sqrt{100} = 10$ (since $x > 0$).

Therefore the perimeter of the triangle is $10 + 8 + 6 = 24$ cm and so the perimeter of the square is also 24 cm.

Since the 4 sides of the square are equal in length, then each measures $\frac{24}{4} = 6$ cm. Thus, the area of the square is $6 \times 6 = 36$ cm^2.

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6. A line segment joins the points $P(-4,1)$ and $Q(1,-11)$. What is the length of PQ?
(A) 13 (B) 12 (C) 12.5 (D) 13.6 (E) 12.6

Source: 2019 Gauss Grade 8 #19

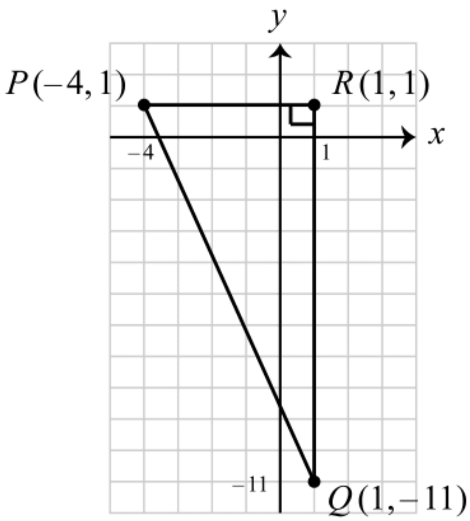
Primary Topics: Geometry and Measurement

Secondary Topics: Coordinate Geometry | Measurement | Pythagorean Theorem

Answer: A

Solution:

The horizontal line through P intersects the vertical line through Q at $R(1,1)$.



Joining P, Q, R creates right-angled $\triangle PQR$, with hypotenuse PQ.
The x-coordinate of P is -4 and the x-coordinate of R is 1, and so PR has length $1-(-4)=5$ (since P and R have equal y-coordinates).
The y-coordinate of Q is -11 and the y-coordinate of R is 1, and so QR has length $1-(-11)=12$ (since Q and R have equal x-coordinates).
Using the Pythagorean Theorem, $PQ^2 = PR^2 + QR^2$ or $PQ^2 = 5^2 + 12^2 = 25 + 144 = 169$, and so $PQ = \sqrt{169} = 13$ (since $PQ > 0$).
(Alternatively, we could have drawn the vertical line through P and the horizontal line through Q which meet at $(-4,-11)$.)

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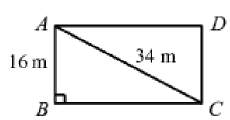
7. Rectangle $ABCD$ has side length $AB = 16$ m and diagonal length $AC = 34$ m, as shown. The perimeter of rectangle $ABCD$ is



(A) 46 m (B) 126 m (C) 100 m (D) 92 m (E) 240 m

Source: 2020 Gauss Grade 8 #14

Primary Topics: Geometry and Measurement

Secondary Topics: Perimeter | Pythagorean Theorem

Answer: D

Solution:

In $\triangle ABC$, $\angle ABC = 90^\circ$ and so by the Pythagorean Theorem, $BC^2 = 34^2 - 16^2 = 1156 - 256$ or $BC^2 = 900$ and so $BC = \sqrt{900} = 30$ m. The perimeter of $ABCD$ is $16 + 30 + 16 + 30 = 92$ m.

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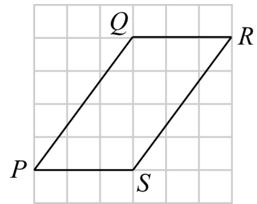
8. In the diagram, points P, Q, R, and S are at intersections of gridlines in a 6×6 grid.



What is the perimeter of parallelogram $PQRS$?

(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

Source: 2022 Pascal Grade 9 #11

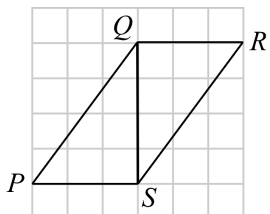
Primary Topics: Geometry and Measurement

Secondary Topics: Coordinate Geometry | Perimeter | Polygons | Graphs | Pythagorean Theorem

Answer: C

Solution:

Since the given grid is 6×6, the size of each of the small squares is 1×1.
This means that $QR = PS = 3$.
Join Q to S.

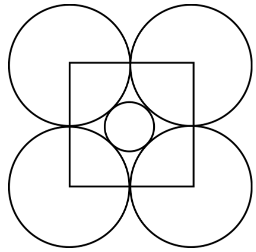


Since QS is vertical, and QR and PS are both horizontal, then $\angle RQS = 90^\circ$ and $\angle PSQ = 90^\circ$.
We note further that $QS = 4$.
Since $\triangle RQS$ is right-angled at Q, by the Pythagorean Theorem,

$$RS^2 = QR^2 + QS^2 = 3^2 + 4^2 = 25$$

Since $RS > 0$, then $RS = 5$.
Similarly, $PQ = 5$.
Thus, the perimeter of $PQRS$ is $PQ + QR + RS + PS = 5 + 3 + 5 + 3 = 16$.

9. Four larger circles with radius 5 are arranged so that their centres are the vertices of a square. Each of the larger circles is tangent to (that is, just touches) two of the other circles, as shown.



A smaller circle with radius r is drawn in the region between the four larger circles. The smaller circle is tangent to each of the larger circles. The value of r is closest to

(A) 1.9 (B) 2.0 (C) 2.1 (D) 2.2 (E) 2.3

Source: 2023 Pascal Grade 9 #20

Primary Topics: Geometry and Measurement

Secondary Topics: Circles | Quadrilaterals | Measurement | Pythagorean Theorem

Answer: C

Solution:

Draw one of the diagonals of the square. The diagonal passes through the centre of the square.
By symmetry, the centre of the smaller circle is the centre of the square. (If it were not the centre of the square, then one of the four larger circles would have to be different from the others somehow, which is not true.) Further, the diagonals of the square pass through the points where the smaller circle is tangent to the larger circles. (The line segment from each vertex of the square to the centre of the smaller circle passes through the point of tangency. These four segments are equal in length and meet at right angles since the diagram can be rotated by 90 degrees without changing its appearance. Thus, each of these is half of a diagonal.) Since each of the larger circles has radius 5, the side length of the square is $5+5=10$. Since the square has side length 10, its diagonal has length $\sqrt{10^2+10^2}=\sqrt{200}$ by the Pythagorean Theorem. Therefore, $5+2r+5=\sqrt{200}$ which gives $2r=\sqrt{200}-10$ and so $r\approx 2.07$. Of the given choices, r is closest to 2.1, or (C).

10. Equilateral triangle ABC has sides of length 4. The midpoint of BC is D, and the midpoint of AD is E. The value of EC^2 is
(A) 7 (B) 6 (C) 6.25 (D) 8 (E) 10

Source: 2022 Gauss Grade 8 #20

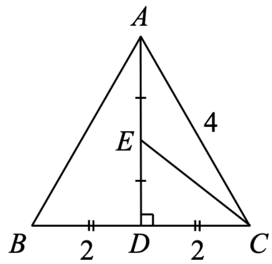
Primary Topics: Geometry and Measurement

Secondary Topics: Triangles | Expressions | Measurement | Pythagorean Theorem

Answer: A

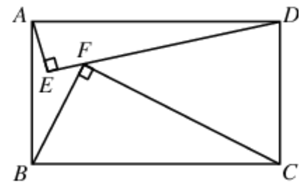
Solution:

Since $\triangle ABC$ is equilateral and has sides of length 4, then $AB = BC = AC = 4$.
The midpoint of BC is D and so $BD = CD = 2$.
The midpoint of AD is E and so $AE = ED$.
Since $AB = AC$ and D is the midpoint of BC, then AD is perpendicular to BC, as shown.



Triangle ADC is a right-angled triangle, and so by the Pythagorean Theorem, we get $(AC)^2 = (AD)^2 + (DC)^2$ or $4^2 = (AD)^2 + 2^2$, and so $(AD)^2 = 16 - 4 = 12$.
Similarly, $\triangle EDC$ is right-angled, and so by the Pythagorean Theorem, we get $(EC)^2 = (ED)^2 + (DC)^2$ or $(EC)^2 = (ED)^2 + 2^2$.
Since $ED = \frac{1}{2}AD$, then $(ED)^2 = \frac{1}{2}AD \times \frac{1}{2}AD$ or $(ED)^2 = \frac{1}{4}(AD)^2$.
Since $AD^2 = 12$, then $(ED)^2 = \frac{1}{4} \times 12 = 3$.
Substituting, we get $(EC)^2 = 3 + 2^2$, and so $(EC)^2 = 7$.

11. In the diagram, right-angled triangles AED and BFC are constructed inside rectangle $ABCD$ so that F lies on DE. If $AE = 21$, $ED = 72$ and $BF = 45$, what is the length of AB?



(A) 50 (B) 48 (C) 52 (D) 54 (E) 56

Source: 2005 Pascal Grade 9 #25

Primary Topics: Geometry and Measurement

Secondary Topics: Triangles | Pythagorean Theorem

Answer: A

Solution:

By the Pythagorean Theorem in $\triangle AED$, $AD^2 = AE^2 + ED^2 = 21^2 + 72^2 = 5625$, so $AD = 75$.
Since $ABCD$ is a rectangle, $BC = AD = 75$. Also, by the Pythagorean Theorem in $\triangle BFC$, $FC^2 = BC^2 - BF^2 = 75^2 - 45^2 = 3600$, so $FC = 60$.
Draw a line through F parallel to AB, meeting AD at X and BC at Y.
To determine the length of AB, we can find the lengths of FY and FX. Step 1: Calculate the length of FY
The easiest method to do this is to calculate the area of $\triangle BFC$ in two different ways.
We know that $\triangle BFC$ is right-angled at F, so its area is equal to $\frac{1}{2}(BF)(FC)$ or~$\frac{1}{2}(45)(60) = 1350$.
Also, we can think of FY as the height of $\triangle BFC$, so its area is equal to $\frac{1}{2}(FY)(BC)$ or $\frac{1}{2}(FY)(75)$.
Therefore, $\frac{1}{2}(FY)(75) = 1350$, so $FY = 36$.

(We could have also approached this by letting $FY = h$, $BY = x$ and so $YC = 75 - x$. We could have then used the Pythagorean Theorem twice in the two little triangles to create two equations in two unknowns.)

Since $FY = 36$, then by the Pythagorean Theorem,

$$BY^2 = BF^2 - FY^2 = 45^2 - 36^2 = 729$$

so $BY = 27$.
Thus, $YC = BC - BY = 48$.
Step 2: Calculate the length of FX
Method 1 -- Similar triangles
Since $\triangle AED$ and $\triangle FXD$ are right-angled at E and X respectively and share a common angle D, then they are similar.
Since $YC = 48$, then $XD = 48$.
Since $\triangle AED$ and $\triangle FXD$ are similar, then $\frac{FX}{XD} = \frac{AE}{ED}$ or $\frac{FX}{48} = \frac{21}{72}$ so $FX = 14$.

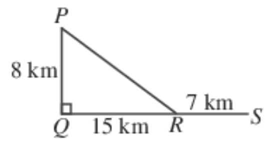
Method 2 -- Mimicking Step 1
Drop a perpendicular from E to AD, meeting AD at Z. We can use exactly the same argument from Step 1 to calculate that $EZ = \frac{504}{25}$ and that $ZD = \frac{1728}{25}$.
Since DF is a straight line, then the ratio $FX : EZ$ equals the ratio $DX : DZ$, ie.~$\frac{FX}{\frac{504}{25}} = \frac{48}{\frac{1728}{25}}$ or $\frac{FX}{504} = \frac{48}{1728}$ or $FX = 14$.

Method 3 -- Areas
Join A to F. Let $FX = x$ and $EF = a$. Then $FD = 72 - a$. Since $AE = 21$ and $ED = 72$, then the area of $\triangle AED$ is $\frac{1}{2}(21)(72) = 756$.
Now, the area of $\triangle AED$ is equal to the sum of the areas of $\triangle AEF$ and $\triangle AFD$, or

$$756 = \frac{1}{2}(21)(a) + \frac{1}{2}(75)(x)$$

so $21a + 75x = 1512$ or $a + \frac{25}{7}x = 72$.
Now in $\triangle FXD$, we have $FX = x$, $GD = 48$ and $FD = 72 - a$.
By the Pythagorean Theorem, $x^2 + 48^2 = (72 - a)^2 = (\frac{25}{7}x)^2 = \frac{625}{49}x^2$.
Therefore, $48^2 = \frac{576}{49}x^2$, or $x = 14$. Therefore, $AB = XY = FX + FY = 36 + 14 = 50$.

12. Asafa ran at a speed of 21 km/h from P to Q to R to S, as shown. Florence ran at a constant speed from P directly to R and then to S. They left P at the same time and arrived at S at the same time. How many minutes after Florence did Asafa arrive at point R?



(A) 0 (B) 8 (C) 6 (D) 7 (E) 5

Source: 2007 Pascal Grade 9 #22

Primary Topics: Geometry and Measurement

Secondary Topics: Triangles | Measurement | Rates | Pythagorean Theorem

Answer: E

Solution:

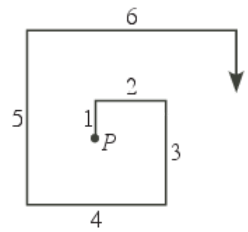
By the Pythagorean Theorem, $PR = \sqrt{QR^2 + PQ^2} = \sqrt{15^2 + 8^2} = \sqrt{289} = 17$ km.
Asafa runs a total distance of $8 + 15 + 7 = 30$ km at 21 km/h in the same time that Florence runs a total distance of $17 + 7 = 24$ km.
Therefore, Asafa's speed is $\frac{30}{24} = \frac{5}{4}$ of Florence's speed, so Florence's speed is $\frac{4}{5} \times 21 = \frac{84}{5}$ km/h.

Asafa runs the last 7 km in $\frac{7}{21} = \frac{1}{3}$ hour, or 20 minutes.
Florence runs the last 7 km in $\frac{7}{\frac{84}{5}} = \frac{35}{84} = \frac{5}{12}$ hour, or 25 minutes.
Since Asafa and Florence arrive at S together, then Florence arrived at R 5 minutes before Asafa.

13. Starting at point P, Breenah constructs a straight sided spiral so that:

- all angles are 90°
- after starting with a line segment of length 1, each side is 1 longer than the previous side.

After completing the side with length 21, Breenah's distance from her original starting point P will be between



(A) 13 and 14 (B) 14 and 15 (C) 15 and 16 (D) 16 and 17 (E) 17 and 18

Source: 2009 Gauss Grade 8 #24

Primary Topics: Geometry and Measurement | Number Sense

Secondary Topics: Measurement | Patterning/Sequences/Series | Pythagorean Theorem

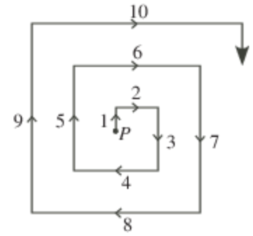
Answer: B

Solution:

Solution 1

Breenah travels along each of the sides in a direction that is either up, down, right or left.

The ``up" sides occur every fourth segment, thus they have lengths $1, 5, 9, 13, 17, 21, \ldots$ or lengths that are one more than a multiple of 4. As we see, the segment of length 21 is an up side.



We see that the upper endpoint of each up segment is 2 units to the left and 2 units above the upper endpoint of the previous up segment. Thus, when Breenah is standing at the upper endpoint of the up segment of length 21, she is $2+2+2+2+2=10$ units to the left and $1+2+2+2+2+2=11$ units above P. The following paragraphs prove this in a more formal~way.

We now determine the horizontal distance from point P to the up side of length 21. Following the spiral outward from P, the first horizontal line segment moves right 2, or $+2$, where the positive sign indicates movement to the right.

The second horizontal segment moves left 4, or -4, where the negative sign indicates movement to the left.

After these two horizontal movements, we are at the line segment of length 5, an up side. To get there, we moved a horizontal distance of $(+2)+(-4)=-2$ or 2 units to the left.

The horizontal distance from P to the next up side (length 9), can be found similarly.

Beginning on the segment of length 5, we are already at -2 or 2 units left of P and we move right 6 (or $+6$), then left 8 (or -8).

Thus, to reach the up side with length 9, we have moved horizontally
$(-2)+(+6)+(-8)=-4$ or 4 units left of P.

This pattern of determining the horizontal distances from P to each of the up sides is continued in the table below.

We now determine the vertical distance from point P to the upper endpoint of the up side with length 21.

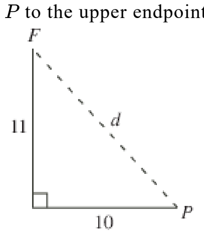
From P, the first vertical segment moves up 1 or $+1$, the second moves down 3 or -3 and the third moves up 5 or $+5$.

Therefore the vertical position of the endpoint of the up side with length 5 is
$(+1)+(-3)+(+5)=+3$ or 3 units above P.

Similarly, we can calculate the vertical position of each of the up side endpoints relative to P and have summarized this in the table below.

Side Length	Horizontal Distance	Vertical Distance
5	$(+2)+(-4)=-2$	$(+1)+(-3)+(+5)=+3$
9	$(-2)+(+6)+(-8)=-4$	$(+3)+(-7)+(+9)=+5$
13	$(-4)+(+10)+(-12)=-6$	$(+5)+(-11)+(+13)=+7$
17	$(-6)+(+14)+(-16)=-8$	$(+7)+(-15)+(+17)=+9$
21	$(-8)+(+18)+(-20)=-10$	$(+9)+(-19)+(+21)=+11$

We now calculate the distance, d, from P to the upper endpoint, F, of the up side of length 21.



Since the calculated distances are horizontal and vertical, we have created a right angle and may find the required distance using the Pythagorean Theorem.

Then $d^2=10^2+11^2=100+121=221$ or $d=\sqrt{221}\approx 14.866$.

Solution 2

If we place the spiral on an xy-plane with point P at the origin, the coordinates of the key points reveal a pattern.

Side Length	Endpoint Coordinates
1	$(0, 1)$
2	$(2, 1)$
3	$(2, -2)$
4	$(-2, -2)$
5	$(-2, 3)$
6	$(4, 3)$
7	$(4, -4)$
8	$(-4, -4)$
⋮	⋮

From the table we can see that after finishing a side having length that is a multiple of 4, say $4k$, we are at the point $(-2k, -2k)$ (the basis for this argument is shown in solution 1).
Therefore, after completing the side of length 20, we are at the point $(-10, -10)$.
All sides of length $4k$ travel toward the left.
We must now move vertically upward 21 units from this point $(-10, -10)$.
Moving upward, this last side of length 21 will end at the point $F(-10, 11)$.
This point is left 10 units and up 11 units from $P(0, 0)$.
Using the Pythagorean Theorem, $PF^2 = 10^2 + 11^2 = 100 + 121 = 221$ or $PF = \sqrt{221} \approx 14.866$.

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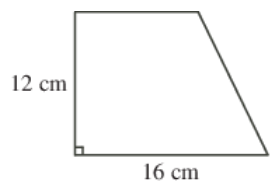
14. The trapezoid shown has a height of length 12 cm, a base of length 16 cm, and an area of 162 cm^2. The perimeter of the trapezoid is



(A) 51 cm (B) 52 cm (C) $49.\overline{6}$ cm (D) 50 cm (E) 56 cm

Source: 2011 Gauss Grade 8 #23

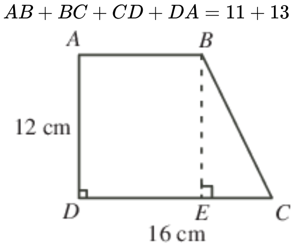
Primary Topics: Geometry and Measurement

Secondary Topics: Quadrilaterals | Area | Perimeter | Pythagorean Theorem

Answer: B

Solution:

We first label the trapezoid $ABCD$ as shown in the diagram below.
Since AD is the perpendicular height of the trapezoid, then AB and DC are parallel.
The area of the trapezoid is $\frac{AD}{2} \times (AB + DC)$ or $\frac{12}{2} \times (AB + 16)$ or $6 \times (AB + 16)$.
Since the area of the trapezoid is 162, then $6 \times (AB + 16) = 162$ and $AB + 16 = \frac{162}{6}$ or $AB + 16 = 27$, so $AB = 11$.
Construct a perpendicular from B to E on DC.
Since AB is parallel to DE and both AD and BE are perpendicular to DE, then $ABED$ is a rectangle.
Thus, $DE = AB = 11$, $BE = AD = 12$, and $EC = DC - DE = 16 - 11 = 5$.
Since $\angle BEC = 90^\circ$, then $\triangle BEC$ is a right-angled triangle.
Thus by the Pythagorean Theorem, $BC^2 = BE^2 + EC^2$ or $BC^2 = 12^2 + 5^2$ or $BC^2 = 169$ so $BC = 13$ (since $BC > 0$).
The perimeter of the trapezoid is $AB + BC + CD + DA = 11 + 13 + 16 + 12 = 52$ cm.



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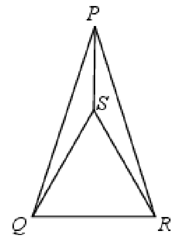
15. In the diagram, $\triangle PQR$ is isosceles with $PQ = PR = 39$ and $\triangle SQR$ is equilateral with side length 30. The area of $\triangle PQS$ is closest to



(A) 68 (B) 75 (C) 50 (D) 180 (E) 135

Source: 2016 Pascal Grade 9 #23

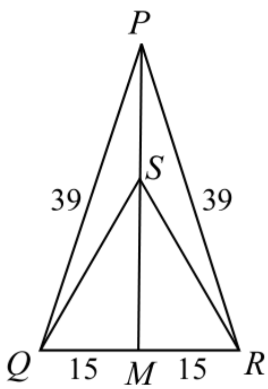
Primary Topics: Geometry and Measurement

Secondary Topics: Area | Triangles | Pythagorean Theorem

Answer: B

Solution:

Join S to the midpoint M of QR.



Since $\triangle SQR$ is equilateral with side length 30, then $QM = MR = \frac{1}{2}QR = 15$.
Since $\triangle SQR$ is equilateral, then SM is perpendicular to QR.
Since $\triangle PQR$ is isosceles with $PQ = PR$, then PM is also perpendicular to QR.
Since PM is perpendicular to QR and SM is perpendicular to QR, then PM and SM overlap, which means that S lies on PM.
By the Pythagorean Theorem,

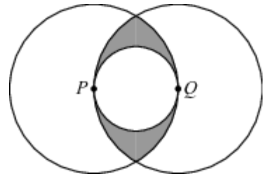
$$PM = \sqrt{PQ^2 - QM^2} = \sqrt{39^2 - 15^2} = \sqrt{1521 - 225} = \sqrt{1296} = 36$$

By the Pythagorean Theorem,

$$SM = \sqrt{SQ^2 - QM^2} = \sqrt{30^2 - 15^2} = \sqrt{900 - 225} = \sqrt{675} = 15\sqrt{3}$$

Therefore, $PS = PM - SM = 36 - 15\sqrt{3}$.
Since QM is perpendicular to PS extended, then the area of $\triangle PQS$ is equal to $\frac{1}{2}(PS)(QM)$.
(We can think of PS as the base and QM as the perpendicular height.)
Therefore, the area of $\triangle PQS$ equals $\frac{1}{2}(36 - 15\sqrt{3})(15) \approx 75.14$.
Of the given answers, this is closest to 75.

16. In the diagram, two larger circles with radius 1 have centres P and Q. Also, the smaller circle has diameter PQ. The region inside the two larger circles and outside the smaller circle is shaded.



The area of the shaded region is closest to

(A) 0.36 (B) 0.38 (C) 0.40 (D) 0.42 (E) 0.44

Source: 2018 Pascal Grade 9 #23

Primary Topics: Geometry and Measurement

Secondary Topics: Area | Circles | Pythagorean Theorem

Answer: E

Solution:

The circle with centre P has radius 1 and passes through Q.
This means that $PQ = 1$.
Therefore, the circle with diameter PQ has radius $\frac{1}{2}$ and so has area $\pi\left(\frac{1}{2}\right)^2 = \frac{1}{4}\pi$.
To find the area of the shaded region, we calculate the area of the region common to both circles and subtract the area of the circle with diameter PQ.
Suppose that the two circles intersect at X and Y.
Join X to Y, P to Q, P to X, P to Y, Q to X, and Q to Y (Figure 1).
By symmetry, the area of the shaded region on each side of XY will be the same.
The area of the shaded region on the right side of XY equals the area of sector $PXQY$ of the left circle minus the area of $\triangle PXY$ (Figure 2).

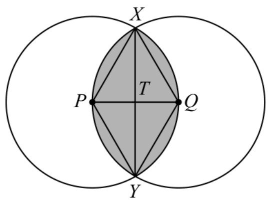


Figure 1

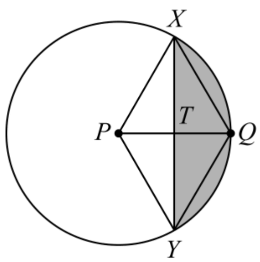


Figure 2

Since each of the large circles has radius 1, then $PQ = PX = PY = QX = QY = 1$.
This means that each of $\triangle XPQ$ and $\triangle YPQ$ is equilateral, and so $\angle XPQ = \angle YPQ = 60^\circ$.
Therefore, $\angle XPY = 120^\circ$, which means that sector $PXQY$ is $\frac{120^\circ}{360^\circ} = \frac{1}{3}$ of the full circle, and so has area $\frac{1}{3}\pi 1^2 = \frac{1}{3}\pi$.
Lastly, consider $\triangle PXY$.
Note that $PX = PY = 1$ and that $\angle XPQ = \angle YPQ = 60^\circ$.
Since $\triangle PXY$ is isosceles and PQ bisects $\angle XPY$, then PQ is perpendicular to XY at T and $XT = TY$.
By symmetry, $PT = TQ$. Since $PQ = 1$, then $PT = \frac{1}{2}$.
By the Pythagorean Theorem in $\triangle PTX$ (which is right-angled at T),

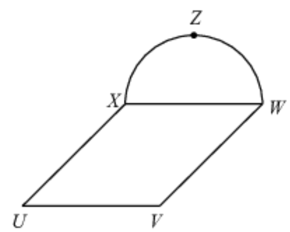
$$XT = \sqrt{PX^2 - PT^2} = \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \sqrt{1 - \frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

since $XT > 0$.
Therefore, $XY = 2XT = \sqrt{3}$.
The area of $\triangle PXY$ equals $\frac{1}{2}(XY)(PT) = \frac{1}{2}(\sqrt{3})\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{4}$.
Now, we can calculate the area of the shaded region to the right of XY to be $\frac{1}{3}\pi - \frac{\sqrt{3}}{4}$, the difference between the area of sector $PXQY$ and the area of $\triangle PXT$.
Therefore, the area of the shaded region with the circle with diameter PQ removed is

$$2\left(\frac{1}{3}\pi - \frac{\sqrt{3}}{4}\right) - \frac{1}{4}\pi = \frac{2}{3}\pi - \frac{\sqrt{3}}{2} - \frac{1}{4}\pi = \frac{5}{12}\pi - \frac{\sqrt{3}}{2} \approx 0.443$$

Of the given choices, this is closest to 0.44.

17. In the diagram, $UVWX$ is a rectangle that lies flat on a horizontal floor. A vertical semi-circular wall with diameter XW is constructed. Point Z is the highest point on this wall. If $UV = 20$ and $VW = 30$, the perimeter of $\triangle UVZ$ is closest to



(A) 95 (B) 86 (C) 102 (D) 83 (E) 92

Source: 2017 Pascal Grade 9 #22

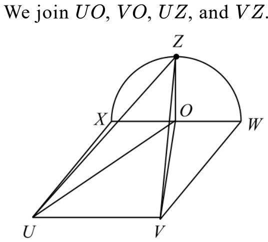
Primary Topics: Geometry and Measurement

Secondary Topics: Measurement | Perimeter | Pythagorean Theorem

Answer: B

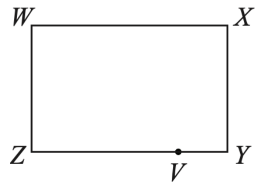
Solution:

The perimeter of $\triangle UVZ$ equals $UV + UZ + VZ$.
We know that $UV = 20$. We need to calculate UZ and VZ.
Let O be the point on XW directly underneath Z.
Since Z is the highest point on the semi-circle and XW is the diameter, then O is the centre of the semi-circle.
We join UO, VO, UZ, and VZ.



Since $UVWX$ is a rectangle, then $XW = UV = 20$ and $UX = VW = 30$.
Since XW is a diameter of the semi-circle and O is the centre, then O is the midpoint of XW and so $XO = WO = 10$.
This means that the radius of the semi-circle is 10, and so $OZ = 10$ as well.
Now $\triangle UXO$ and $\triangle VWO$ are both right-angled, since $UVWX$ is a rectangle.
By the Pythagorean Theorem, $UO^2 = UX^2 + XO^2 = 30^2 + 10^2 = 900 + 100 = 1000$ and $VO^2 = VW^2 + WO^2 = 30^2 + 10^2 = 1000$.
Each of $\triangle UOZ$ and $\triangle VOZ$ is right-angled at O, since the semi-circle is vertical and the rectangle is horizontal.
Therefore, we can apply the Pythagorean Theorem again to obtain $UZ^2 = UO^2 + OZ^2$ and $VZ^2 = VO^2 + OZ^2$.
Since $UO^2 = VO^2 = 1000$, then $UZ^2 = VZ^2 = 1000 + 10^2 = 1100$ or $UZ = VZ = \sqrt{1100}$.
Therefore, the perimeter of $\triangle UVZ$ is $20 + 2\sqrt{1100} \approx 86.332$.
Of the given choices, this is closest to 86.

18. Rectangle $WXYZ$ has $WX = 4$, $WZ = 3$, and $ZV = 3$.



The rectangle is curled without overlapping into a cylinder so that sides WZ and XY touch each other. In other words, W touches X and Z touches Y. The shortest distance from W to V through the inside of the cylinder can be written in the form $\sqrt{\frac{a + b\pi^2}{c\pi^2}}$ where a, b and c are positive integers. The smallest possible value of $a + b + c$ is

(A) 12 (B) 26 (C) 18 (D) 19 (E) 36

Source: 2021 Pascal Grade 9 #23

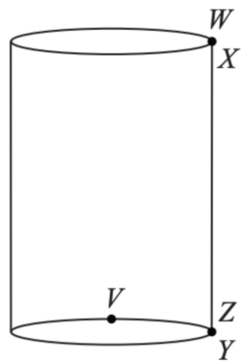
Primary Topics: Geometry and Measurement

Secondary Topics: Measurement | Cylinders | Pythagorean Theorem

Answer: C

Solution:

When the cylinder is created, W and X touch and Z and Y touch.



This means that WY is vertical and so is perpendicular to the plane of the circular base of the cylinder.

This means that $\triangle VYW$ is right-angled at Y.

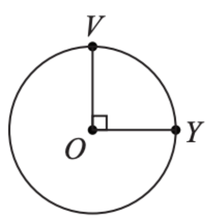
By the Pythagorean Theorem, $WV^2 = WY^2 + VY^2$.

Note that WY equals the height of the rectangle, which is 3 (the length of WZ) and that VY is now measured *through* the cylinder, not along the line segment ZY.

Let O be the centre of the circular base of the cylinder.

In the original rectangle, $ZY = WX = 4$ and $ZV = 3$, which means that $VY = 1 = \frac{1}{4}ZY$.

This means that V is one-quarter of the way around the circumference of the circular base from Y back to Z.



As a result, $\angle YOV = 90^\circ$, since 90° is one-quarter of a complete circular angle.

Thus, $\triangle YOV$ is right-angled at O.

By the Pythagorean Theorem, $VY^2 = VO^2 + OY^2$.

Since YO and OV are radii of the circular base, then $VO = OY$ and so $YV^2 = 2VO^2$.

Since the circumference of the circular base is 4 (the original length of ZY), then if the radius of the base is r, we have $2\pi r = 4$ and so $r = \dfrac{4}{2\pi} = \dfrac{2}{\pi}$.

Since $VO = r$, then $YV^2 = 2VO^2 = 2\left(\dfrac{2}{\pi}\right)^2 = \dfrac{8}{\pi^2}$.

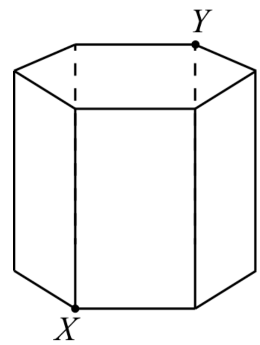
This means that

$$WV^2 = WY^2 + YV^2 = 9 + \frac{8}{\pi^2} = \frac{9\pi^2 + 8}{\pi^2} = \frac{8 + 9\pi^2}{\pi^2}$$

and so $WV = \sqrt{\dfrac{8 + 9\pi^2}{1 \cdot \pi^2}}$.

Since the coefficient of π^2 in the denominator is 1, it is not possible to "reduce" the values of a, b and c any further, and so $a = 8$, $b = 9$, and $c = 1$, which gives $a + b + c = 18$.

19. A hexagonal prism has a height of 165 cm. Its two hexagonal faces are regular hexagons with sides of length 30 cm. Its other six faces are rectangles.



A fly and an ant start at point X on the bottom face and travel to point Y on the top face. The fly flies directly along the shortest route through the prism. The ant crawls around the outside of the prism along a path of constant slope so that it winds around the prism exactly $n + \frac{1}{2}$ times, for some positive integer n. The distance crawled by the ant is more than 20 times the distance flown by the fly. What is the smallest possible value of n?

Source: 2022 Pascal Grade 9 #25

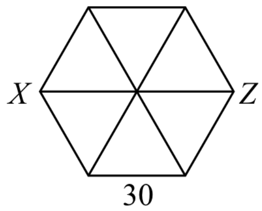
Primary Topics: Geometry and Measurement

Secondary Topics: Prisms | Rates | Measurement | Pythagorean Theorem

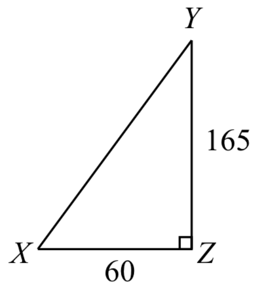
Answer: 19

Solution:

Throughout this solution, we remove the units (cm) as each length is in these same units.
First, we calculate the distance flown by the fly, which we call f.
Let Z be the point on the base on the prism directly underneath Y.



Since the hexagonal base has side length 30, then $XZ = 60$.
This is because a hexagon is divided into 6 equilateral triangles by its diagonals, and so the length of the diagonal is twice the side length of one of these triangles, which is twice the side length of the hexagon. Also, $\triangle XZY$ is right-angled at Z, since XZ lies in the horizontal base and YZ is vertical.

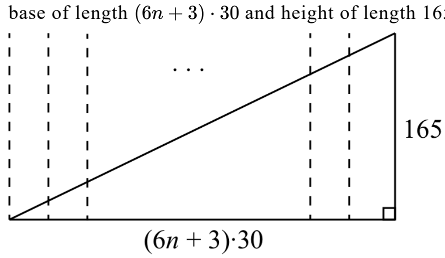


By the Pythagorean Theorem, since $XY > 0$, then

$$XY = \sqrt{XZ^2 + YZ^2} = \sqrt{60^2 + 165^2}$$

Therefore, $f = XY = \sqrt{60^2 + 165^2}$.

Next, we calculate the distance crawled by the ant, which we call a.
Since the ant crawls $n + \frac{1}{2}$ around the prism and its crawls along all 6 of the vertical faces each time around the prism, then it crawls along a total of $6(n + \frac{1}{2}) = 6n + 3$ faces.
To find a, we "unwrap" the exterior of the prism.
Since the ant passes through $6n + 3$ faces, it travels a "horizontal" distance of $(6n + 3) \cdot 30$. Since the ant moves from the bottom of the prism to the top of the prism, it passes through a vertical distance of 165.
Since the ant's path has a constant slope, its path forms the hypotenuse of a right-angled triangle with base of length $(6n + 3) \cdot 30$ and height of length 165.



By the Pythagorean Theorem, since $a > 0$, then $a = \sqrt{((6n + 3) \cdot 30)^2 + 165^2}$.

Now, we want a to be at least $20f$. In other words, we want to find the smallest possible value of n for which $a > 20f$.
Since these quantities are positive, the inequality $a > 20f$ is equivalent to the inequality $a^2 > 20^2 f^2$.
The following inequalities are equivalent:

$$
\begin{aligned}
a^2 &> 20^2 f^2 \\
((6n+3)\cdot 30)^2 + 165^2 &> 400(60^2 + 165^2) \\
(6n+3)^2 \cdot 30^2 + 165^2 &> 400(60^2 + 165^2) \\
(6n+3)^2 \cdot 2^2 + 11^2 &> 400(4^2 + 11^2) && \text{(dividing both sides by } 15^2\text{)} \\
4(6n+3)^2 + 121 &> 400 \cdot 137 \\
4(6n+3)^2 &> 54\,679 \\
(6n+3)^2 &> \frac{54\,679}{4} \\
6n+3 &> \sqrt{\frac{54\,679}{4}} && \text{(since both sidesare positive)} \\
6n &> \sqrt{\frac{54\,679}{4}} - 3 \\
n &> \frac{1}{6}\left(\sqrt{\frac{54\,679}{4}} - 3\right) \approx 18.986
\end{aligned}
$$

Therefore, the smallest positive integer n for which this is true is $n = 19$.

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20. Points $A(-3,5)$, $B(0,7)$ and $C(r,t)$ lie along a line. If $BC = 4AB$ and $r > 0$, what is the value of $r+t$?

Source: 2025 Cayley Grade 10 #21

Primary Topics: Geometry and Measurement

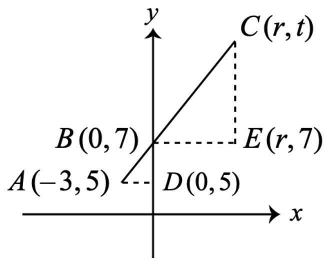
Secondary Topics: Coordinate Geometry | Equations Solving | Pythagorean Theorem

Answer: 27

Solution:

Solution 1:

Plot the points A, B, and C as well as $D(0,5)$ and $E(r,7)$ so that $\triangle ABD$ and $\triangle BCE$ have right angles at D and E, respectively.
These two right-angled triangles each have a horizontal side and a vertical side, as shown.



Since A, B, and C are all on the same line, AD is parallel to BE, and BD is parallel to CE, we must have that $\angle DAB = \angle EBC$ and $\angle DBA = \angle ECB$.
Hence, $\triangle ABD$ is similar to $\triangle BCE$.
Using common ratios, we get $\dfrac{BC}{AB} = \dfrac{EC}{DB} = \dfrac{BE}{AD}$.
Each of EC and DB is vertical, so their lengths are the difference between the y-coordinates of the two points.
Thus, $EC = t - 7$ and $DB = 7 - 5 = 2$.
Similarly, the lengths of BE and AD are each the different between the x-coordinates of the two points, giving $BE = r$ and $AD = 3$.
It is given that $BC = 4AB$, so $\dfrac{BC}{AB} = 4$. Therefore, $4 = \dfrac{EC}{DB} = \dfrac{t-7}{2}$ and $4 = \dfrac{BE}{AD} = \dfrac{r}{3}$.
Rearranging these two equations gives $8 = t - 7$ or $t = 15$ and $12 = r$, so $r + t = 27$.

Solution 2:

Using the distance formula,

$$AB = \sqrt{(7-5)^2 + (0-(-3))^2} = \sqrt{4+9} = \sqrt{13}$$

and

$$BC = \sqrt{(t-7)^2 + (r-0)^2} = \sqrt{(t-7)^2 + r^2}$$

It is given that $BC = 4AB$, so $4\sqrt{13} = \sqrt{(t-7)^2 + r^2}$. Squaring both sides gives $16 \times 13 = (t-7)^2 + r^2$.

It is also given that A, B, and C are on a common line. This implies that the slope of the segment AB is the same as the slope of the segment BC.
These slopes are $\dfrac{7-5}{0-(-3)} = \dfrac{2}{3}$ and $\dfrac{t-7}{r-0} = \dfrac{t-7}{r}$, respectively.
Setting the computed slopes equal, we have $\dfrac{t-7}{r} = \dfrac{2}{3}$, or $t - 7 = \dfrac{2}{3}r$.
Substituting into $16 \times 13 = (t-7)^2 + r^2$, we get the following equivalent equations.

$$\begin{aligned} 16 \times 13 &= \left(\frac{2}{3}r\right)^2 + r^2 \\ 16 \times 13 &= \frac{4}{9}r^2 + r^2 \\ 16 \times 13 &= \frac{13}{9}r^2 \\ 16 \times 9 &= r^2 \\ \sqrt{16}\sqrt{9} &= \sqrt{r^2} \\ 12 &= r \end{aligned}$$

where the final equality is because r is assumed to be positive.
Therefore, we have $r = 12$, from which we get $t - 7 = \dfrac{2}{3} \times 12 = 8$, or $t = 15$.
The answer to the question is $r + t = 12 + 15 = 27$.